Combinatorial Counting

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Motivation

“Counting” problems show up in many different aspects of statistics and computer science:

- Networking & Cryptology, error retrieval and information encoding
- Optimizing Algorithms, discrete formulas over computational solutions
- Graph Theory, understanding relations between objects
- Complexity Theory, understanding program runtimes

Rule of Product

A procedure can be broken down into a sequence of two tasks. If there are x ways to do *Task 1* **and** y ways to do *Task 2:*

There are $x \cdot y$ ways to do Task 1 and Task 2

Q: I want to know how many lunch combinations I can make at Knowlton Café. If there are 3 entrees and 8 drinks I like, how many different lunches can I buy? $\mathbf{(3 \cdot 8)}$

Q2: What if I add one of 5 different bags of chips? $\mathbf{(24 \cdot 5)}$

Q3: In a 12-player game, how many ways can a gold, silver, and bronze medal be awarded? $\mathbf{12 \cdot 11 \cdot 10}$

Rule of Sum

A procedure can be broken down into two separate tasks. If there are x ways to do *Task 1* **or** y ways to do *Task 2:*

There are $x + y$ ways to do *Task 1* or *Task 2*

Q: The MCS department is selecting a representative for a new committee, they can select one person. How many choices are there if there are 24 professors and 93 students?

$$\mathbf{24 + 93}$$

Sum and Product as Sets

If $A_1, A_2, \ldots, A_m$ are sets, where each element is a possible choice, then the number of possible selections is choosing one element from the Cartesian product all the sets:

Product: $|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_m|$

If $B_1, B_2, \ldots, B_n$ are sets, where each element is a possible choice, then the number of ways to choose one element from any B_i , given all sets are disjoint, is:

Sum: $|B_1 \cup B_2 \cup \cdots \cup B_n| = |B_1| + |B_2| + \ldots + |B_n|$ when $B_i \cap B_j = \emptyset, \ \forall(i, j)$

Q: For the rule of sum, why do the sets have to be disjoint?

Avoid double-counting

Counting IP Addresses

Any device connected on the internet has an IP (Internet Protocol) address. Most commonly known are IPv4, and IPv6.

IPv4 conforms to the following:

[0 – 255] . [0 – 255] . [0 – 255] . [0 – 255]

Q: How many IPv4 addresses are possible?

$256^4 \approx 4.3$ billion distinct, possible

Internet Protocol v. 6

IPv6 was created to solve the anxiety of IP address exhaustion. An IPv6 address is of the following form:

8 groups, each group made of 4 hexadecimal numbers, each hexadecimal is 4 bits

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Example: AB01::9872::123F::AE98::D152::34FC::0023::356E
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Q: How many different overall numbers in just one group of 4 hexadecimal numbers?

$\mathbf{16^4}$

Q: How many different overall IPv6 addresses?

$\mathbf{(16^4)^8 = 16^{32} = 3.4 \cdot 10^{38} \approx 340}$ **trillion trillion trillion addresses**

Bit String Practice

Bit strings are a sequence (or `string`), where each character is either 0 or 1.

Practice Problems:

- Write out the different possible bit strings of length 3.

000 001 010 100 011 101 111

- Using the rule of product, how many bit strings of length 10 exist?

- How many bit strings of length n?

2^n

- Bob likes their bit strings using $\{0, 1\}$ as possible letters, but Alice uses $\{a, b\}$ as their bit string alphabet. Using Bob's or Alice's alphabet, how many bit strings exist of length 15?

$2^{15} + 2^{15}$

Decision Trees

Counting problems can (typically) be visualized as trees. Start at the root, each branch represents a possible choice for a task. Each layer is a different task.

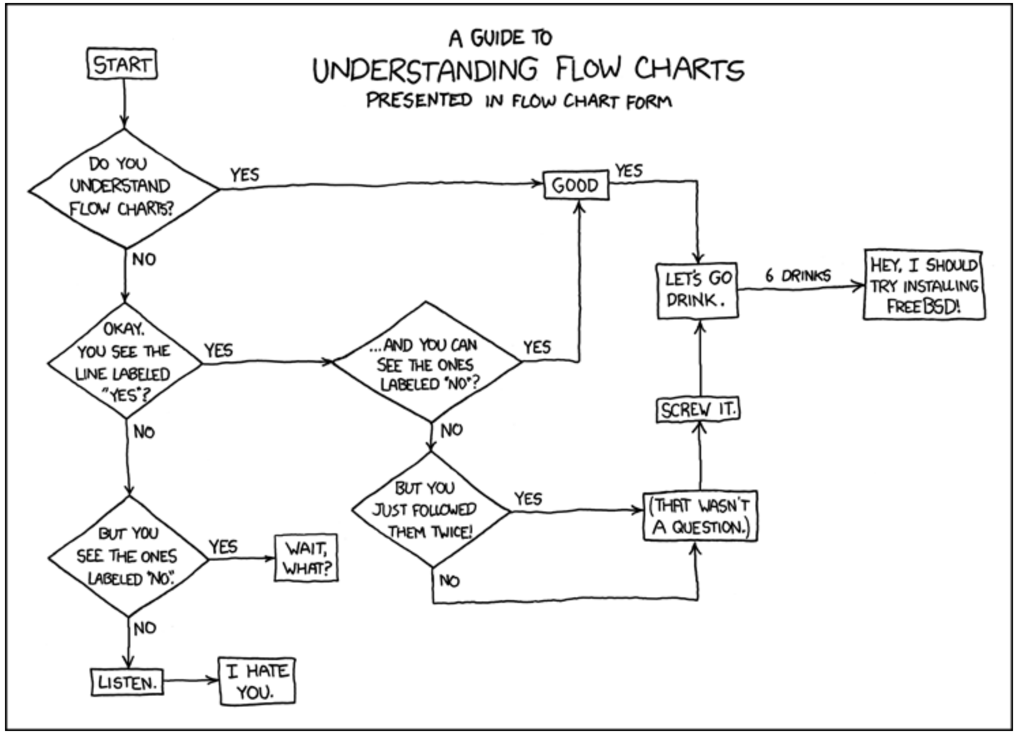
Kind of like a more structured **flow chart.**

Q: Back to the lunch analogy:

Task 1: choose an entrée from {salad, vegetarian, sandwich}

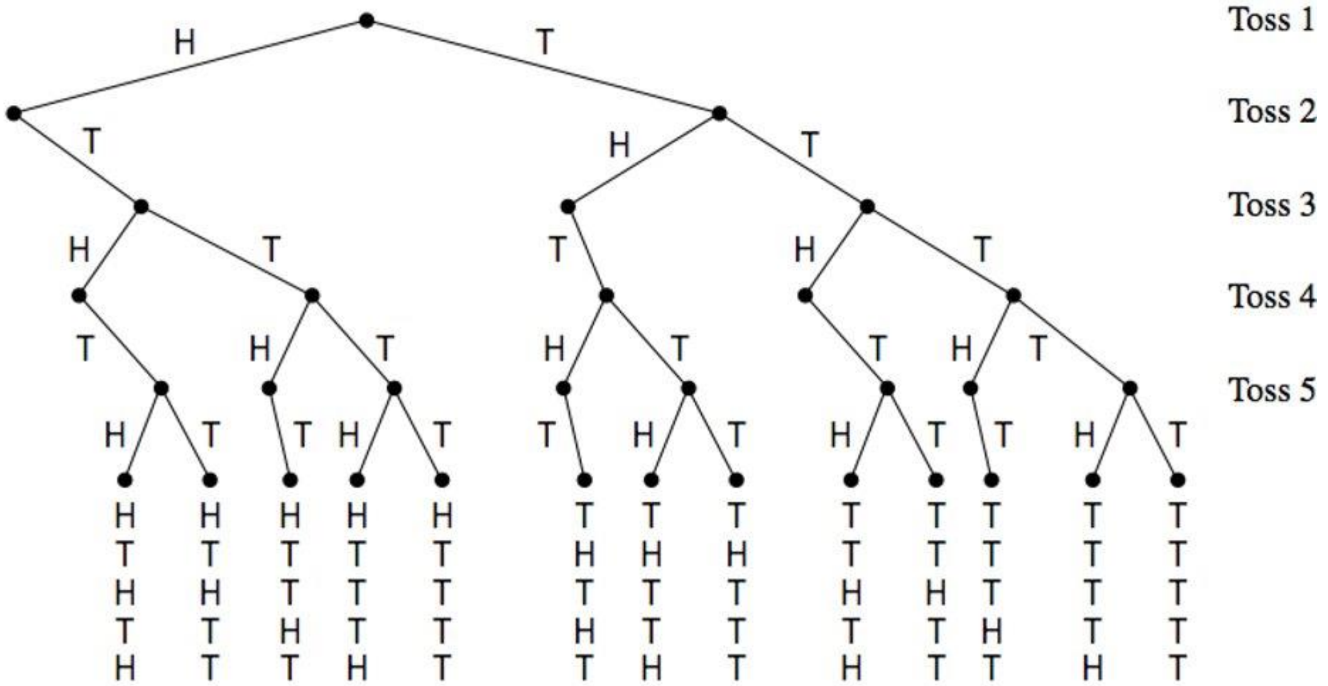
Task 2: choose a drink from {hot tea, coffee, smoothie}

However, I do not like smoothies with a salad, and I do not like hot tea with a sandwich. How many different lunches will I be happy with?



Decision Trees Continued

Q: If I flip a coin 5 times, how many ways can I flip my coin so that I do not get heads twice in a row?



Inclusion & Exclusion

Q: Suppose there are 30 engineers. 15 are comfortable programming in JavaScript, 12 are comfortable in PHP, and 7 are comfortable in both. How many engineers are not comfortable with JavaScript or PHP?

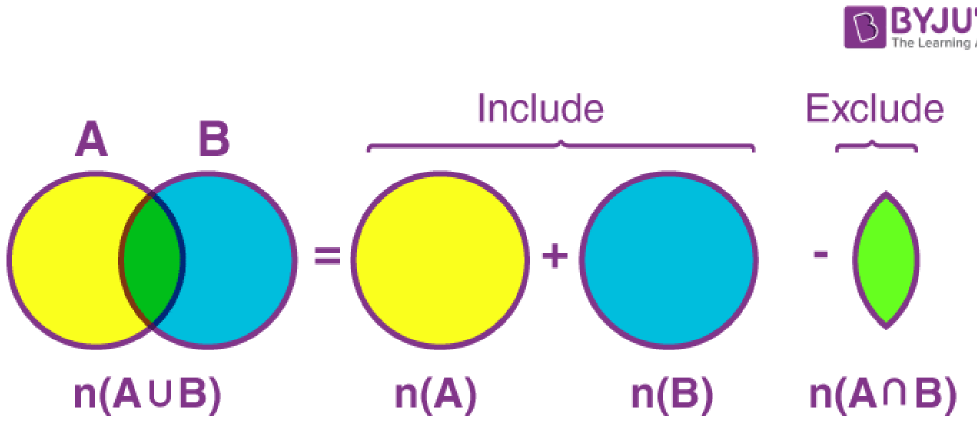
$\mathbf{30 - [(15 - 7) + 7 + (12 - 7)] = 30 - 20 = 10}$ **engineers**

or $\mathbf{30 - [15 + 12 - 7]}$

Principle of Inclusion and Exclusion

If a task can be done either in one of x ways or in one of y ways, then the total number of ways to do the task is $x + y$ minus the number of ways to do the task that are common to the two different ways:

$$|A \cup B| = |A| + |B| - |A \cap B|$$



I/E Practice

Practice:

- How many bit-strings of length 8 end with `00` or start with a `1`? $\mathbf{2^7 + 2^6 - 2^5}$

Challenges:

- If $|A \cup B| = |A| + |B| - |A \cap B|$ is the formula for two sets, what would the Venn diagram and formula look like for 3 sets?

- If there are 31 students in this class, how many possible study groups exist with 3 or more students?

All possible groups – [empty group + single group + double group]

$2^{31} - \left[1 + 31 + \frac{31 \cdot 30}{2}\right] = 2147483151$